The Permeability Variogram from Pressure Transients of Multiple Wells: Theory and 1-D Application

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This paper presents a new approach for the estimation of the large-scale correlation function of reservoir heterogeneity from the analysis of the pressure transient response of multiple wells. The approach is based on the theory of small fluctuations of the logarithm of the permeability and models the response of the "ensemble-average pressure", obtained by averaging the pressure response of multiple well tests. A non-local diffusivity equation for the ensemble-average pressure, which incorporates directly the permeability correlation function, is obtained, from the analysis of which the permeability semi-variogram of the reservoir can be constructed in principle. We consider a particular application to 1-D geometries, where a type-curve is also derived for the case of an exponential correlation function. Numerical simulations results support the applicability of the method.

The stochastic representation of subsurface reservoirs requires reliable estimates of the correlation structure of reservoir attributes. Basing the correlation structure on core or well log derived data requires a substantial number of probing points, each of which has a limited radius of investigation. Single- or multiple- well pressure transient tests offer a substantially larger investigation capability, while requiring fewer probing points. At present, however, a theoretical foundation to relate the pressure transient responses to heterogeneity characteristics, in general, and to the nature of correlation structure, in particular, is missing.

The application of well-testing to idealized homogeneous reservoirs has been extensive and it is described in many classical works. The classical approach provides a first approximation to of the flow properties, such as the average permeability. With the increasing recognition of the heterogeneous nature of oil reservoirs, however, many attempts have recently been made to extend the usefulness of well testing to the interpretation of the heterogeneity of reservoir properties.

The origin of most of the works in this area can be traced to Oliver, who evaluated the next, beyond the homogeneous, term in the asymptotic expansion of the well pressure response for a system with a weakly varying permeability. Oliver showed that the problem effectively reduces to that of flow in a reservoir with an effective permeability which is radial dependent. Oliver's solution is expressed in terms of a composite integral, involving the product of the unknown radially-averaged permeability fluctuation with a known kernel (similar to an integral transform), the inversion of which could in principle yield the radially-averaged permeability fluctuations. In a subsequent paper, Oliver argued against the existence of a unique solution when matching this solution with a discrete well response. He then, proposed an approximation based on the method of Backus and Gilbert, which provides a smooth approximation to the true solution. Feitosa et al. proposed a different approach, based on an Inverse Solution Algorithm (ISA), which although not guaranteeing uniqueness in all cases, gives practically useful results and appears to circumvent some of the problems.

Subsequent work built on the same line of reasoning. Oliver extended his approach to problems involving storativity (in addition to transmissibility) fluctuations, and proceeded with the development of an analogous equation for interference testing. Yadavalle et al. proceeded one step further, and used the results of Feitosa et al. to infer a semi-variogram based on the inducted data of the radially-averaged fluctuations.

In this paper, we consider a similar problem but with a somewhat different perspective. As in Oliver et al., we assume a permeability field with small fluctuations. The permeability field is assumed random but stationary (namely gradients in its statistical properties do not exist). We consider both storativity and transmissibility, although for simplicity, the applications will be restricted to heterogeneous transmissibility only. The small fluctuations approximation is equivalent to assuming small fluctuations in the logarithm of the permeability. This assumption is commonly made in many stock-
astic studies in reservoir engineering. Thus, the validity of this theory has the same range as that of previous studies in this field\(^1\). However, in contrast to the previous work, we consider the response of the ensemble-average pressure, namely the average pressure over many realizations of the stochastic permeability field. In practice, the ensemble-average pressure can be approximately obtained as the arithmetic mean of the pressure response from tests in \(N\) different and non-interfering wells (and where \(N\) is sufficiently large). Our method is based on stochastic averaging and it is similar to the approach of Gelhar and Axness\(^6\) for obtaining the dispersion coefficient in tracer dispersion in a heterogeneous reservoir.

We show the following: (i) The ensemble-average pressure in a heterogeneous reservoir does not satisfy the homogeneous equation with constant coefficients, as one might intuitively expect; (ii) from inverting the ensemble-average response at the well, one can obtain the true semi-vario-gam of the permeability field. Furthermore, based on a 1-D application, we show that: (iii) the ensemble-average pressure at the well initially follows the homogeneous response, but approaches at large times the response of a random field (with a different average permeability), the transition depending on the correlation length (see also Dagan\(^3\)). In the latter cases and for the case of an exponential correlation, we also derive a type-curve to match the pressure derivative data, which allows the simultaneous estimation of the variance and the correlation length.

The paper is organized as follows: We first present the small-fluctuation theory and obtain an integrodifferential equation for the ensemble-average pressure. This equation is then solved recursively (in an asymptotic expansion similar to Oliver\(^1\)) in an infinite 1-D reservoir with a well producing at its center. Illustrative examples for simple cases, such as exponential correlation functions, are provided from which type-curves are constructed. Some numerical experiments of the pressure response in heterogeneous reservoirs with variable variance and correlation structures are also presented. The theory and experiments are compared and found to be in generally good agreement. In addition, an integral equation relating the well response to the semi-vario-gam of the permeability field is also derived and discussed. Finally, a brief discussion of the 2-D case, which is currently under investigation, is also given.

Theory

The pressure diffusivity equation in an isotropic, heterogeneous medium reads as follows

\[
\Phi(X) \frac{\partial p}{\partial t} = \nabla \cdot \left( \frac{K(X)}{\mu c} \nabla p \right)
\]

(1)

where \(\mu\) denotes viscosity, \(c\) is compressibility, the porosity, \(\Phi\), and the permeability, \(K\), are stationary stochastic variables, and capital letters denote dimensional variables. We shall use a general notation, although the particular emphasis is on the production from a single well at the constant volume flow rate \(Q\).

To simplify the notation, we introduce the dimensionless space variable \(x = X/l\), where \(l\) is a characteristic length (e.g. the well radius, \(r_w\)), the dimensionless time \(t = \frac{tK}{l^2\mu c}\), and the dimensionless pressure

\[
p = \frac{2xH_k}{Q}\left( p - p_{1} \right)
\]

We also rescale porosity, \(\phi = \Phi/\bar{\Phi}\), and permeability, \(k = K/\bar{K}\), using the respective mean values

\[
\bar{\phi} = \langle\phi\rangle \quad \text{and} \quad \ln\bar{K} = \langle\lnK\rangle
\]

(2)

where brackets denote ensemble averages, namely averages over (infinitely) many realizations. The assumption of a log-normal distribution in permeability and of the geometric mean for the permeability implied in (2) is commonly made in many problems in reservoir engineering (e.g. see Gelhar and Axness\(^8\)). Then, Equation (1) reads

\[
\phi(x) \frac{\partial p}{\partial t} = \nabla \cdot (k(x)\nabla p)
\]

(3)

We seek the solution of (3) subject to appropriate initial and boundary conditions. In particular, in this paper we are concerned with problems where the initial pressure is constant and the reservoir is unbounded, thus we take \(p(0, x) = 0\) and \(p(t, \infty) = 0\).

To solve (3) it is convenient to work in Laplace space. Taking the Laplace transform in time of (3) we obtain

\[
s\phi(x)\hat{p} = \nabla \cdot (k(x)\nabla \hat{p})
\]

(4)

where \(s\) is the Laplace variable, superscript “hat” denotes Laplace transform and we have omitted subscript \(x\). Equation (4) can be further rearranged in the form

\[
g(x)\hat{p} = \nabla^2 \hat{p} + f(x)\nabla \hat{p}
\]

(5)

where we defined the two heterogeneity functions

\[
g(x) \equiv \frac{\phi(x)}{k(x)}
\]

(6)

and

\[
f(x) \equiv \lnk(x)
\]

(7)

expressing storativity and transmissibility, respectively. We will consider the approximate solution of (5) in the case of a heterogeneous with small fluctuations.

1. Theory of small fluctuations. Denoting fluctuations by superscript prime, we will take \(k = 1 + k'\), \(\phi = 1 + \phi'\), \(g = 1 + g'\) and \(\hat{p} = <\hat{p}> + \hat{p}'\). By definition, we have \(f \equiv f' \approx k'\) and \(<f' > = <g' > = <\hat{p}' > = 0\). However, to second-order in fluctuations, we have \(<g' > = <f'^2/2 + f'\phi' + \cdots > \equiv m \neq 0\) and \(<k' > = <f'^2/2 + \cdots > \equiv n \neq 0\). Then, equation (5) becomes

\[
s(1 + m)(<\hat{p}> + <g'\hat{p}'>) = \nabla^2 <\hat{p}> + <\nabla f' \cdot \nabla \hat{p}'>
\]

(8)

Next, we take the ensemble average of (8), to \(g'^{-1}\) following equation

\[
\left<s(1 + m)(<\hat{p}> + <g'\hat{p}'>)\right> = \nabla^2 <\hat{p}> + <\nabla f' \cdot \nabla \hat{p}'>
\]

(9)

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We note that the equation satisfied by the ensemble-average pressure is not the standard diffusivity equation, which in Laplace space reads as
\[ s < \hat{p} > = -\nabla^2 < \hat{p} > \tag{10} \]
and which one normally solves in the homogeneous case, but it also includes terms from the fluctuations. This is an important first result of the paper. A similar conclusion was reached previously by other authors in related contexts (e.g. Gavalias and Yortsos\textsuperscript{10}, Goddard\textsuperscript{11}, Noetinger and Gautier\textsuperscript{12}). It is possible, therefore, to obtain information on the fluctuations and their variance, by solving equation (9).

For the boundary conditions, we focus on the specific problem of drawdown from a single well. For simplicity, we will assume a cylindrical geometry, although all the results to be obtained apply, with some simple modifications, to other geometries as well. In the notation of the rescaled variables, the boundary condition for the constant drawdown at the well reads
\[ -\int_0^{2\pi} \frac{\partial < \hat{p} >}{\partial r} \bigg|_{r = r_{wd}} (1 + f'(r_{wd}, \theta) + \cdots) d\theta - \int_0^{2\pi} \frac{\partial \hat{p}}{\partial r} \bigg|_{r = r_{wd}} (1 + f'(r_{wd}, \theta) + \cdots) d\theta = \frac{2\pi}{s} \tag{11} \]
where we introduce the radial coordinates \( r \) and \( \theta \), and we also denoted \( r_{wd} \equiv r_w / l \) (where \( l \) is the characteristic length scale). Assuming hydrostatic equilibrium, the pressure at the wellbore must be independent of the angular coordinate \( \theta \), hence we also have
\[ \frac{\partial \hat{p}(r_{wd}, \theta)}{\partial \theta} = 0 \tag{12} \]
Ensemble averaging (11) and (12) gives the boundary conditions for the ensemble-average pressure \( < \hat{p} > \). To second-order in fluctuations, we have
\[ -(1 + n) \int_0^{2\pi} \frac{\partial < \hat{p} >}{\partial r} \bigg|_{r = r_{wd}} d\theta - \int_0^{2\pi} f'(r_{wd}, \theta) \frac{\partial \hat{p}}{\partial r} \bigg|_{r = r_{wd}} d\theta = \frac{2\pi}{s} \tag{13} \]
and
\[ \frac{\partial < \hat{p} >}{\partial \theta} \bigg|_{r = r_{wd}} = 0 \tag{14} \]
For the solution of (9) and (13), it is necessary to develop equations for the pressure fluctuation. This is discussed next.

2. The pressure fluctuation. To obtain the partial differential equation and the boundary conditions for the pressure fluctuation we subtract the equations satisfied by the ensemble-average from the full equations (i.e. (9) from (8) and (13) from (11)) and retain only linear (first-order) terms. Such a linearization is common to many problems in heterogeneous media (e.g. see Gelhar and Axness\textsuperscript{8}). We then obtain
\[ \nabla^2 \hat{p} - s \hat{p} = s(g' - m) < \hat{p} > = -\nabla^2 \hat{p} - s(g' - m) < \hat{p} > \tag{15} \]
To first-order, we may further take \( g' \approx \psi' - f' \) and neglect \( m \), hence
\[ \nabla^2 \hat{p} - s \hat{p} = s\psi' < \hat{p} > - \nabla \psi' \nabla < \hat{p} > \tag{16} \]
This is the basic equation satisfied by the pressure fluctuation.

The corresponding boundary conditions are obtained likewise. After some calculations we obtain at \( r = r_{wd} \)
\[ -\int_0^{2\pi} \frac{\partial \hat{p}}{\partial r} d\theta = \int_0^{2\pi} f'(r_{wd}, \theta) \frac{\partial < \hat{p} >}{\partial r} d\theta \tag{17} \]
and
\[ \frac{\partial \hat{p}(r_{wd}, \theta)}{\partial \theta} = 0 \tag{18} \]
Equation (16) is an inhomogeneous Laplace equation. For its solution with the boundary conditions (17)-(18), we use a Green's function approach, assuming certain mild conditions on the fluctuations. The derivation is shown in Appendix A, where it is found that the pressure fluctuation is
\[ \hat{p}'(\mathbf{x}) = -\int_\Omega \{ G(\mathbf{x} | \mathbf{\xi})(s\psi'(\mathbf{\xi}) < \hat{p} > (\mathbf{\xi}, s) + f'\nabla_\xi^2 < \hat{p} > \}
+ f'\nabla_\xi G \cdot \nabla_\xi < \hat{p} > \} d\xi \tag{19} \]
where \( \Omega \) is the region of integration. Expression (19) can be used in (9) to obtain an equation for the ensemble-average pressure.

3. The ensemble-average pressure. Introducing the pressure fluctuation results into equation (9) we can perform the required ensemble averaging. Details are given in Appendix B. After several calculations we obtain the following equation satisfied by the ensemble-average pressure
\[ \nabla^2 < \hat{p} > - s(1 + m) < \hat{p} > = \int_\Omega \{ \nabla_\xi^2 < \hat{p} > - s < \hat{p} > \} [s(C_f(\mathbf{\xi}) - C_{sf}(\mathbf{\xi}))] G(\mathbf{x} | \mathbf{\xi}) \]
\[ + C_f(\mathbf{\xi}) \mathbf{\xi}^T \nabla_\xi G + \int_\Omega [ < \hat{p} > - s^2(C_{sf}(\mathbf{\xi}))]
- C_{\phi \phi}(\mathbf{\xi}) G(\mathbf{x} | \mathbf{\xi}) + C_{\phi \xi}(\mathbf{\xi}) \mathbf{\xi}^T \nabla_\xi G \]
\[ + \nabla_\xi^2 < \hat{p} > + [s(C_f(\mathbf{\xi}) - C_{sf}(\mathbf{\xi}))] \nabla_\xi G \]
\[ + C_f(\mathbf{\xi}) \mathbf{\xi}^T \nabla_\xi (\nabla_\xi G)^T \} d\xi \tag{20} \]
where \( C_{ij} \) denotes the covariance of the variables \( i \) and \( j \), the superscript \( T \) denotes the transpose, and we also defined \( \zeta = \mathbf{x} - \mathbf{\xi} \). This integrodifferential equation describes to second-order in fluctuations the evolution of the ensemble-average pressure in the Laplace space. With the exception of the terms containing derivatives in the covariance terms and the mixed covariances, this
equation is similar to an equation derived previously by Oliver⁵, as discussed below.

For the remainder of this paper, we will mainly neglect variations in porosity and consider only those in permeability. Then, equation (20) simplifies to the final form

\[
\nabla^2 \hat{p} = s(m + n) \nabla^2 \hat{p} \\
= \int \frac{(\nabla^2 \hat{p})}{s} \nabla \hat{G} \xi \\
+ C_{ff}(\xi) \frac{s}{s} \nabla \hat{G} \xi \\
+ \int \nabla^2 \hat{p} \cdot s C_{ff}(\xi) \nabla \hat{G} \xi \\
+ C_{ff}(\xi) \frac{s}{s} \nabla \hat{G} \xi \\
\]

Equations (20) and (21) also hold for the corresponding 1-D problem, where the domain of integration is \((-\infty, \infty)\) and a well at \(x = 0\) is producing at constant rate.

4. A recursive solution. Solving (20) or (21) is not a trivial task. A simpler approach would be to take an asymptotic expansion in terms of the variance of the fluctuations, which has already assumed small in the derivation. In this approach, one takes

\[
\nabla^2 < \hat{p}_{m+1} > + \nabla^2 < \hat{p}_{m+2} > + \cdots
\]

where the second term in this expansion is assumed of the same order as the variance of the fluctuations. Then, it is straightforward to show that \(\hat{p}_{m+2} > \) satisfies the homogeneous Laplace equation (10), the solution of which, subject to the boundary conditions (13) and (14), is well known (see below for some examples).

In the case when both storage and transmissibility fluctuations are considered, the next term, \(\hat{p}_{m+2} > \), of the expansion can be shown to satisfy the following equation

\[
\nabla^2 < \hat{p}_{m+2} > = s m < \hat{p}_{m+2} > + \\
\int \frac{(\nabla^2 \hat{p})}{s} \nabla \hat{G} \xi \\
+ C_{ff}(\xi) \frac{s}{s} \nabla \hat{G} \xi \\
- C_{ff}(\xi) \nabla \hat{G} + C_{ff}(\xi) \frac{s}{s} \nabla \hat{G} \xi \\
\]

This equation has some similarity to Oliver’s⁶, when only transmissibility fluctuations are considered, equation (23) further simplifies to

\[
\nabla^2 < \hat{p}_{m+2} > = s m < \hat{p}_{m+2} > + \\
\int \nabla^2 \hat{p} \cdot s C_{ff}(\xi) \nabla \hat{G} \xi \\
\]

This is the second important result of the paper. It provides an equation for \(\hat{p}_{m+2} > \), the solution of which, in conjunction with \(\hat{p}_{m+1} > \), gives the behavior of the ensemble-average pressure. More importantly, however, it can be shown that the final solution can be expressed in terms of an integral involving only the semi-variogram \(C_{ff}(\xi)\), and not its derivatives. This is significant as the resulting equation can be inverted to yield the semi-variogram. Specific expressions are given below.

Application to 1-D Geometries

In 1-D geometries in an infinite interval with a well at \(x = 0\), the the Green’s function and the leading-order term for the pressure response can be easily computed. We have

\[
G_{1D}(\xi) = \frac{1}{2\sqrt{\pi}} \exp(-|\xi|/\sqrt{\pi})
\]

and

\[
< \hat{p}_{0} > (x) = \frac{1}{2\sqrt{\pi}} \exp(-|x|/\sqrt{\pi})
\]

In this case, the evaluation of the solution of (24) is straightforward, but quite elaborate and will not be presented here. A detailed derivation is available from the authors by request. In the next section we only present the final results obtained.

1. The ensemble-average pressure at the well.

After a considerable amount of calculations, we obtain the following simple expression for the ensemble-average of the fluctuations at the well \((x = 0)\)

\[
< p_{2} > (0, t) = \frac{m}{2} \sqrt{\frac{t}{\pi}} + \int_{0}^{\infty} C_{ff}(\rho) L(\frac{\rho}{\sqrt{\pi}}) d\rho
\]

where we introduced the kernel function

\[
L(z) = \frac{1}{4\sqrt{\pi}} z \exp(-z^2) - \frac{3}{8} \frac{\pi}{\sqrt{\pi}}
\]

A plot of the function \(L(z)\) is shown in Fig. 1. We observe that this function has a maximum at \(\sqrt{\pi}\), and decays to zero at large \(z\). Combining (27) with the inverse of (26), the overall solution in real space is to second-order in fluctuations,

\[
< p > (0, t) = \left[1 + \frac{\sigma^2}{4}\right] \sqrt{\frac{t}{\pi}} + \\
\int_{0}^{\infty} C_{ff}(\rho) L(\frac{\rho}{\sqrt{\pi}}) d\rho
\]

where \(\sigma^2\) is the variance. Given \(< p > (0, t)\) data, this equation can be inverted to yield the semi-variogram \(C_{ff}(\rho)\). Before we address this, it is interesting to consider the behavior of (29) at the two limiting cases of small and large times, respectively.

At small times, the integral on the RHS of (29) is proportional to \(\sqrt{t}\), namely
Fig. 1 – The kernel function \( L(z) \).

\[
\int_0^\infty C_{ff}(\rho) L\left(\frac{\rho}{\sqrt{t}}\right) \, d\rho = \sqrt{t} \int_0^\infty C_{ff}(z) L(z) \, dz = \frac{\sigma^2 \sqrt{t}}{4\sqrt{\pi}} \tag{30}
\]

where we used \( \int_0^\infty L(z) \, dz = 1/(4\sqrt{\pi}) \). It follows that the early-time behavior of the ensemble pressure response is that of the homogeneous

\[
< p > (0, t) = \sqrt{\frac{t}{\pi}} \tag{31}
\]

but with average permeability the geometric average \( \bar{K} \).

At large times, on the other hand, the integral tends to the constant

\[
\text{const} = \int_0^\infty C_{ff}(\rho) L(0) \, d\rho = \frac{3}{8} \int_0^\infty C_{ff}(\rho) \, d\rho \tag{32}
\]

and the pressure response is that of the random field problem, namely

\[
< p > (0, t) \sim \left(1 + \frac{\sigma^2}{4}\right) \sqrt{\frac{t}{\pi}} + \text{const} \tag{33}
\]

Thus, the ensemble-average response for the uncorrelated, random problem has the same shape as the homogeneous, but a different prefactor. In the classical theory, therefore, such behavior would be interpreted as corresponding to a homogeneous system, but with a different average permeability

\[
\bar{K}_{\text{rand}} = \frac{\bar{K}}{\left(1 + \frac{\sigma^2}{4}\right)^{\frac{1}{2}}} \tag{34}
\]

where we have converted (34) to a dimensional notation. We note that, to first-order in the variance, (34) is the same expression for the steady-state effective permeability of a 1-D random system

\[
\bar{K}_{\text{eff}} = \bar{K} \left(1 - \frac{\sigma^2}{2}\right) \tag{35}
\]

derived by Dagan. The transition between the two cases depends on the correlation length, as illustrated below for an exponential correlation function.

2. Exponential correlation function. For the case of an exponential correlation function of the form \( C(\rho) = \sigma^2 \exp\left(-\frac{\rho}{\lambda}\right) \), where \( \lambda \) is the correlation length, the above simplifies further to

\[
< p > (0, t) = \left(1 + \frac{\sigma^2}{4}\right) \sqrt{\frac{t}{\pi}} + \sigma^2 \lambda J \left(\frac{\sqrt{t}}{2\lambda}\right) \tag{36}
\]

where \( J(w) \) denotes the expression

\[
J(w) = \frac{w}{8} \left(\frac{2}{\sqrt{\pi}} - 2w \exp w^2 \text{erfc} w\right) + \frac{3}{8} (-1 + \exp w^2 \text{erfc} w) \tag{37}
\]

We note that \( J(0) = 0 \), \( J(\infty) = -\frac{3}{8} \) and \( J'(0) = -\frac{1}{8} \).

A plot of (36) is shown in Fig. 2 for various values of the correlation length. Shown are the pressure response (Fig. 2a) and its modified derivative, \( \sqrt{\pi t} \frac{dC_{str}}{dt} \) (Fig. 2b). We note the following: At small values of time, all curves approach the homogeneous solution (where \( \sigma = 0 \)). As time increases, however, the solutions depart from the homogeneous, the departure being faster at smaller values of \( \lambda \) and asymptotically approach the random case, where \( \lambda = 0 \). From (36), the latter is

\[
< p >_{\text{rand}} (0, t) \sim \left(1 + \frac{\sigma^2}{4}\right) \sqrt{\frac{t}{\pi}} + \text{const} \tag{38}
\]

which confirms the previous general result, equation (33).

It is easily checked that the derivative \( \sqrt{\pi t} \frac{dC_{str}}{dt} \) is only a function of the similarity variable \( w = \frac{\sqrt{t}}{2\lambda} \)

\[
\sqrt{\pi t} \frac{d< p >}{dt} = \left(1 + \frac{\sigma^2}{4}\right) \frac{1}{2} + \frac{\sigma^2 \sqrt{\pi}}{4} J'(w) \tag{39}
\]

At small times, the derivative takes the value \( \frac{1}{8} \), while at large times it approaches asymptotically the random case value \( \frac{3}{8} \left(1 + \frac{\sigma^2}{4}\right) \) (see also Fig. 2b). The approach to the random case is faster for smaller values of the correlation length, as one might expect. Indeed, because of the dependence on \( w \) in (39), all curves in Fig. 2b would collapse into a single curve, if plotted vs the similarity variable \( w \).

3. Type-curves for exponential correlations. Rearrangement of (36) allows the introduction of the following type-curve

\[
\frac{< p > (0, t)}{\lambda} = \frac{2}{\sqrt{\pi}} \left(1 + \frac{\sigma^2}{4}\right) w + \sigma^2 J(w) \tag{40}
\]

which is plotted in Fig. 3a for various values of \( \sigma \). Although a matching with the pressure data is in prin-
Fig. 2 – The ensemble-average response at a well for an exponential correlation function for various values of the correlation length: (a) Pressure, (b) modified pressure derivative.

Fig. 3 – Type-curves for an exponential correlation function: (a) Pressure for different values of the variance $\sigma^2$, (b) modified pressure derivative.

principle possible, using (40), perhaps a more convenient type-curve match is obtained using the pressure derivative. From (39) we can see that the latter can be cast in terms of the following type curve

$$8 \sqrt{\pi} \frac{\langle \Delta p \rangle}{\sigma^2} = 1 + 2 \sqrt{\pi} J'(w)$$

(41)

This is plotted in Fig. 3b vs the similarity variable $w$. Thus, under the assumption of an exponentially correlated permeability, this type curve can be used to estimate the variance and the correlation length directly by appropriately shifting the data along the two axes to match the type curve.

In this exercise, the also unknown geometric mean, $\bar{K}$, is to be estimated by matching the early-time data with the homogeneous solution.

A comparison between analytical predictions and numerical results is shown in Fig. 4. The computations were done on a rectilinear uniform grid of size 1001 using finite differences. Ensemble-averages were computed from 120 different realizations. The number of realizations was found to affect the results, the effect becoming less significant, however, as the number exceeded 100. Fig. 4a shows a comparison between the analytical expression (36) and the numerical results (top curve) with a variance equal to 0.741. The agreement is quite good for a range of intermediate time values, but becomes less satisfactory at very early (and also very late times, not shown here), due to numerical effects (finite size effects) and also due to the relatively large variance taken. Numerical results for the derivative are shown in Fig. 4b. This figures should be contrasted with Fig. 2, where analytical results are shown. We notice a qualitative agreement between theory and simulation, although the agreement is less satisfactory at early times.
it involves explicitly the variogram, which for isotropic systems naturally depends on the radial distance only. Its solution can be obtained using the various techniques of integral equations of this type (e.g., see Groetsch\textsuperscript{14}), or by applying the approaches of Oliver\textsuperscript{2} (such as the Backus and Gilbert\textsuperscript{2} method) and Feitosa et al.\textsuperscript{4} Further work in this direction is currently in progress.

### Discussion of the 2-D Case

Consider, next, the case of radial geometry with a single well in an unbounded space. To leading order, the ensemble-average pressure satisfies the well-known solution

\[
< \bar{p}_0 > = \frac{K_0(\sqrt{r})}{s^{3/2}K_1(\sqrt{s})}
\]  

(44)

For the solution of the next term in the expansion, equation (24), we use (44) and a Green's function approach, where the Green's function was defined in Appendix A. However, its computation is not trivial.

1. **Some general results.** After repeated use of Green's theorems and several manipulations (details available from the authors by request), we obtain the following expression for \( < \bar{p}_2 > \)

\[
< \bar{p}_2 > (x) = m \int_\Omega \nabla_x^T \cdot \nabla_x' G(x|x')dx' - \frac{4m\pi}{d} G(x|1) + \int_\Omega \int_\Omega C_{ff}(|x'-\xi|)(F-H)dx'd\xi
\]

where we introduced the auxiliary functions

\[
F(x,x',\xi) = \nabla_x^T G(x|x') \cdot \nabla_x^T \nabla_\xi < \bar{p}_0 > \cdot \nabla_\xi G(x'|\xi)
\]  

(45)

and

\[
H(x,x',\xi) = \nabla_x^T \cdot [G(x|x') \nabla_x^T \nabla_\xi < \bar{p}_0 > \cdot \nabla_\xi G(x'|\xi)]
\]  

(46)

In the above we implicitly made use of the boundary condition for \( < \bar{p}_2 > \), which also involves the pressure fluctuation, assumed without loss \( r_{wd} = 1 \), and made use of the fact that \( \nabla_x(\nabla_\xi G)^T \) is a second-order tensor, and of the relations

\[
\nabla_\xi |\xi| = \frac{\xi}{|\xi|} \quad \text{and} \quad \nabla_\xi |\xi| = -\frac{\xi}{|\xi|^2}
\]  

(47)

The domain \( \Omega \) denotes the entire plane minus the unit circle. It follows that the solution at the well is expressed in terms of a (multiple) integral involving the correlation function. In general notation, and after inversion, the pressure at the well will vary as

\[
< \bar{p}_2 > w = mf(t) + \int_\Omega C_{ff}(|x'-\xi|)f(t,x',\xi)dxd\xi
\]  

(48)
where the functions \( f_1 \) and \( f_2 \) need to be explicitly expressed. This work is currently in progress. Although complex, in general, it should be emphasized that equation (48) only involves the semi-variogram, thus permitting its estimation from the inversion of pressure data. A similar expression, namely one involving only the variogram, also results for other geometries (such as spherical).

2. An alternative approach. Alternatively, a similar expression can be obtained by utilizing Oliver's\(^1\) solution. Before we proceed, we need to state the differences between the two approaches. In Oliver\(^1\), the following expansion is taken for the pressure around a single well

\[ p = p_1 + p_2 + \cdots \]  \hspace{1cm} (49)

where \( p_1 \) is first-order in fluctuations. In the present paper, on the other hand, we take for the ensemble-average pressure (which is the average response of multiple wells) the expansion

\[ <p> = p_0 + <p_2> + \cdots \]  \hspace{1cm} (50)

where \( <p_2> \) is second order in fluctuations (the first order vanishes in the ensemble-average by definition). In essence, in calculating \( p_2 \) in (50) we have obtained the next term in Oliver's expansion.

By comparing (49) and (50) it is apparent that for non-trivial information involving the ensemble-average pressure one should make use of (50). On the other hand, potentially useful results can also be obtained using the ensemble variance of the fluctuations of the single well pressure (or its derivative), which from (49) reads

\[ V(t) \equiv <(p(1,t) - p_0(1,t))^2> = <p_1^2(1,t)> \]  \hspace{1cm} (51)

The RHS above can be calculated using Oliver's\(^1\) expressions. Then, we get the result

\[ V(t) = \int_{\Omega} \frac{C_{1f}(|x - x'|)}{|x||x'|} G\left(\frac{|x|}{\sqrt{t}}\right) G\left(\frac{|x'|}{\sqrt{t}}\right) dx'dx \]  \hspace{1cm} (52)

where the function \( G \) was defined by Oliver\(^1\)

\[ G(\rho) = 1 + \pi \int_{0}^{\infty} \exp\left(-\frac{x^2}{\rho^2}\right) J_1(x)Y_1(x)dx \]  \hspace{1cm} (53)

and \( J_1 \) and \( Y_1 \) denote Bessel functions of the first kind.

Equation (52) (or its equivalent involving the pressure derivatives, which can be readily obtained as above), is functionally the same as (48). Differences and similarities between the two approaches, and further investigation of the 2-D problem are currently under study.

Conclusions

A theoretical foundation to relate the correlation structure of reservoir attributes to the ensemble-average of well transient data has been presented. Using a theory of small fluctuations we developed equations for the ensemble-average pressure to second-order in fluctuations containing the correlation functions. From the solution of this equation the ensemble-average response at the well can be obtained, the inversion of which leads to the estimation of the semi-variogram. Specifically, for an 1-D geometry, it was shown that the ensemble-average pressure follows at early times the behavior of a homogeneous reservoir, but approaches at late times that of a random field, with a reduced average permeability. The time for the transition between these two limits depends on the correlation length. For the simple case of an exponential correlation function, a type-curve for the derivative was also developed. Finally, we obtained an integral equation for the semi-variogram, in terms of the ensemble-average response at the well. Comparison with numerical results confirms the validity of the theory in the cases tried.

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Nomenclature

\[ C = \text{correlation function} \]
\[ c = \text{compressibility} \]
\[ G = \text{Green's function} \]
\[ H = \text{reservoir thickness} \]
\[ K = \text{permeability} \]
\[ k = \text{dimensionless permeability} \]
\[ l = \text{characteristic length} \]
\[ P = \text{pressure} \]
\[ p = \text{dimensionless pressure} \]
\[ Q = \text{volume flow rate} \]
\[ r = \text{radius} \]
\[ s = \text{Laplace variable} \]
\[ T = \text{time} \]
\[ t = \text{dimensionless time} \]
\[ X = \text{space variables} \]
\[ x = \text{dimensionless space variables} \]
\[ \gamma = \text{semi-variogram} \]
\[ \theta = \text{angular coordinate} \]
\[ \lambda = \text{correlation length} \]
\[ \mu = \text{viscosity} \]
\[ \sigma = \text{standard deviation} \]
\[ \Phi = \text{porosity} \]
\[ \phi = \text{rescaled porosity} \]

Subscripts

\[ f = \text{transmissibility} \]
\[ g = \text{storativity} \]
\[ i = \text{initial} \]
\[ w = \text{well} \]
\[ \phi = \text{porosity} \]
where \( \delta(r) \) is the Dirac delta function. For reasons that will become apparent later, we take \( G \) to vanish at infinity and to satisfy the following boundary conditions at the well

\[
\frac{\partial G(r_{wd}, \theta | \xi)}{\partial \theta} = 0 \quad \text{and} \quad \int_0^{2\pi} \frac{\partial G(r, \theta | \xi)}{\partial r} \bigg|_{r=r_{wd}} \ d\theta = 0 \quad (A-2)
\]

This definition of the Green's function differs from the one used by Oliver\(^5\), who used the free-space Green's function (equivalent to a vanishing well radius). The Green's function above is more appropriate for a domain involving a well of finite radius. In terms of this Green's function, the solution of (15) is

\[
p'(x) = \int_0^\Omega G(x | \xi) \left[ -sg'(\xi) < \hat{\rho} > (\xi, s) + \nabla_\xi f^* \cdot \nabla_\xi < \hat{\rho} > \right] d\xi
\]

\[-\int_0^\Omega \left[ G(x | r_{wd}, \alpha) \frac{\partial^2 p'(|\rho, \alpha|)}{\partial \rho} \bigg|_{\rho=r_{wd}} \right] d\alpha \quad (A-3)
\]

where \( \Omega \) is the region of integration, and we introduced the cylindrical coordinates \( \rho \) and \( \alpha \) for the space variable \( \xi \). This equation enables us to compute the terms in (9) containing the pressure fluctuation \( p' \) and to close the scheme, if we also note that the fluctuation is linearly related to the ensemble average pressure. Before doing so, we rewrite (A-3) by making use of Green's theorem (as also done by Oliver) and by noting that the last term on the RHS of (A-3) vanishes, in view of the reciprocity principle, \( G(x | \xi) = G(\xi | x) \), of equation (18) and of the second of equations (A-2). Thus,

\[
p'(x) = -\int_0^\Omega \left[ G(x | \xi) sg'(\xi) < \hat{\rho} > (\xi, s) + f^* \nabla_\xi \cdot (G(x | \xi) \nabla_\xi < \hat{\rho} >) \right] d\xi
\]

\[-\int_0^\Omega G(x | r_{wd}, \alpha) \left( \frac{\partial^2 p'(|\rho, \alpha|)}{\partial \rho} \bigg|_{\rho=r_{wd}} \right) d\alpha \quad (A-4)
\]

However, due to the first of equations (A-2), the function \( G \) in the last integral of (A-4) is independent of \( \alpha \) and can be taken out of the integral. Then, because of the boundary condition (17), the entire surface integral vanishes, and (A-4) further simplifies to

\[
p'(x) = -\int_0^\Omega \left[ G(x | \xi) sg'(\xi) < \hat{\rho} > (\xi, s) + f^* \nabla_\xi < \hat{\rho} > \right] d\xi
\]

\[+ f^* \nabla_\xi G \cdot \nabla_\xi < \hat{\rho} > \quad (A-5)
\]

where, we recall that to this order, \( g' = \phi' - f' \).

---

**Appendix A: Solution of the equation for the pressure fluctuation**

Consider the Green's function for the operator \( \nabla^2 - s \), namely take

\[
\nabla^2 G - sG = -\delta(x - \xi) \quad (A-1)
\]
Appendix B: Ensemble-averaging of pressure fluctuations

Introduction of the expression for the pressure fluctuation into equation (9) allows for the ensemble averaging process to be carried out. We obtain

\[ < g' \hat{p} > = - \int_{\Omega} [G(x|\xi)s < g'(x)g'(|\xi|) > < \hat{p} > (x, s) \]
\[ + < g'(x)f'(\xi) > \nabla_{\xi}^{2} < \hat{p} > \]
\[ + < g'(x)f'(\xi) > \nabla_{\xi}G \nabla_{\xi} < \hat{p} > ] d\xi \quad (B-1) \]

and

\[ < \nabla f' \cdot \nabla \hat{p} > = - \int_{\Omega} \left[ (s < \hat{p} > < g'(\xi) \nabla_{\xi}f'(|\xi|) > x > \nabla_{\xi}G \right] \]
\[ + \nabla_{\xi}^{2} < \hat{p} > < f'(\xi) \nabla_{\xi}f'(|\xi|) > \nabla_{\xi}G \]
\[ + < f'(\xi) \nabla_{\xi}f'(|\xi|) > \cdot \nabla_{\xi}G \nabla_{\xi} < \hat{p} > ] d\xi \quad (B-2) \]

To proceed further, the ensemble-averages in the two integrals must be evaluated. For the isotropic correlation structure assumed here, the following apply (see also Dagan)\(^5\)

\[ < g'(x)g'(|\xi|) > = C_{yy}(|\xi|) \quad (B-3) \]
\[ < g'(x)f'(|\xi|) > = C_{yf}(|\xi|) \quad (B-4) \]
\[ < g'(\xi) \nabla_{\xi}f'(|\xi|) > = \frac{C_{yf}(|\xi|)}{|\xi|} \zeta \quad (B-5) \]

and

\[ < f'(\xi) \nabla_{\xi}f'(|\xi|) > = \frac{C_{ff}(|\xi|)}{|\xi|} \zeta \quad (B-6) \]

where \( C_{ij} \) denotes the covariance of the variables \( i \) and \( j \), and we also defined \( \zeta = x - \xi \). Substitution of the above expressions in (9) gives the final equation satisfied by the ensemble-average pressure

\[ \nabla^{2} < \hat{p} > = -s(1 + m) < \hat{p} > = \]
\[ - \int_{\Omega} [G(x|\xi)s^{2}C_{yy}(|\xi|) < \hat{p} > + sC_{yf}(|\xi|) \nabla_{\xi}G \nabla_{\xi} < \hat{p} > ] d\xi \]
\[ + \int_{\Omega} \left[ (sC_{yf}(|\xi|) < \hat{p} > + \nabla_{\xi}^{2} < \hat{p} > C_{ff}(|\xi|) \frac{c_{T}}{|\xi|} \nabla_{\xi}G \right] \]
\[ + \frac{C_{ff}(|\xi|)}{|\xi|} \left< \nabla_{\xi}G \cdot \nabla_{\xi} < \hat{p} > \right> ] d\xi \quad (B-7) \]

Here the superscript \( T \) denotes the transpose. This integrodifferential equation describes second-order in fluctuations the evolution of the ensemble-average pressure in the Laplace space. By using the definition of \( g' \), we can further express the covariances in terms of \( \phi \) and \( f \). After substitution and some manipulations, we get

\[ \nabla^{2} < \hat{p} > = -s(1 + m) < \hat{p} > = \]
\[ \int_{\Omega} (\nabla_{\xi}^{2} < \hat{p} > - s < \hat{p} > ) \left[ s(C_{ff}(|\xi|) - C_{yf}(|\xi|))G(x|\xi) \right] \]