Simple tests for the validity of correlation function models
on the circle

Tilmann Gneiting

Department of Statistics, University of Washington, Seattle, WA 98112-4322, USA

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Abstract
We present two simple and efficient tests for the positive definiteness of a given function on the circle. The first
criterion is an analogue of Pólya's theorem, and the second is a necessary condition in terms of derivatives. Some hints
at applications in geostatistics are given as well. © 1998 Elsevier Science B.V. All rights reserved

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1. Introduction

Let $\varphi(t)$, $t \in \mathbb{R}$, be the correlation function of a second-order stationary, mean-square continuous,
and real-valued stochastic process. From the Bochner–Khintchine criterion (Yaglom, 1987, p. 103) we know that $\varphi(t)$
has a representation of the form

$$\varphi(t) = \int_{[0,\infty)} \cos(xt) \, dF(x)$$

(1)

for $F$ some distribution function on $[0, \infty)$. Conversely, any function $\varphi(t)$ of the form (1) is positive definite on
the real line with $\varphi(0) = 1$, and therefore a correlation function. Now, let $\psi(t)$, $t \in [-K,K]$, be the correlation
function of a second-order stationary, mean-square continuous, and real-valued process on the circle $[-K,K]$, with
the points $-K$ and $K$ identified. The general form of $\psi(t)$ is the Fourier cosine series

$$\psi(t) = \frac{p_0}{2} + \sum_{n=1}^{\infty} p_n \cos\left(\frac{\pi nt}{K}\right), \quad p_n \geq 0$$

(2)

with only nonnegative coefficients $p_0/2, p_1, p_2, \ldots$ summing to 1 (Yaglom, 1987, pp. 387–388). Thus, any
correlation function on the circle $[-K,K]$ can be continued with period $2K$ to a correlation function on the real line.

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1 Present address: Tilmann Gneiting, Department of Mathematics, Universität Bayreuth, D-95440 Bayreuth, Germany.
Wood (1995) has recently asked for the opposite direction: Suppose \( \varphi(t) \), \( t \in \mathbb{R} \), is a correlation function on the real line. Define

\[
\psi(t) = \varphi(t), \quad t \in [-K,K].
\]  

(3)

Is \( \psi(t) \), defined by (3), positive definite, that is, a correlation function on the circle \([-K,K]\)? The problem is obviously equivalent to the question whether the Fourier cosine coefficients

\[
p_n = \frac{1}{K} \int_{-K}^{K} \psi(t) \cos\left(\frac{\pi nt}{K}\right) \, dt
\]  

(4)

are nonnegative for all integers \( n \geq 0 \). Wood (1995) gives a number of conditions that imply a positive or negative answer, and the intention of this note is to supplement his results. In Section 2, we present an analogue of Pólya’s criterion for functions on the circle, and we hint at applications in geostatistics. Section 3 provides a simple but powerful necessary condition for a positive answer to the above question. Further comments in Section 4 conclude the paper.

It should be evident that Wood’s question is equivalent to a continuation and uniqueness problem for characteristic functions. Let \( \varphi(t) \), \( t \in \mathbb{R} \), be the characteristic function of a symmetric probability distribution on the real line. Is the \( 2K \)-periodic function \( \varphi_0(t) \), which coincides with \( \varphi(t) \) for \( t \in [-K,K] \), also a characteristic function? Dugué and Girault (1955, p. 8) and Lévy (1961, p. 324) considered this problem and thereby established results almost identical to Theorems 2.1 and 2.2 of Wood (1995). See also Theorem 4.3.1, Theorem 4.3.2, and the first footnote on p. 86 of Lukacs (1970). Although expressed in terms of correlation functions, the criteria below apply equally to the continuation problem for characteristic functions. The unifying notion is that of positive definite functions.

2. Pólya’s theorem on the circle

Our first result is an analogue of the celebrated Pólya’s criterion for functions on the real line (Lukacs, 1970, p. 83; Yaglom, 1987, pp. 136–137). Note we need not check a priori whether the candidate function \( \psi(t) \) is a truncated correlation function on the line.

Theorem 1. Suppose the function \( \psi(t) \), defined for \( t \in [-K,K] \), has the following properties:

(i) \( \psi(t) \) is real-valued, even, and continuous,
(ii) \( \psi(0) = 1 \),
(iii) \( \int_{-K}^{K} \psi(t) \, dt \geq 0 \),
(iv) \( \psi(t) \) is nonincreasing and convex for \( t \in [0,K] \).

Then \( \psi(t) \), \( t \in [-K,K] \), is a correlation function on the circle of circumference \( 2K \).

Proof. We need to show that all Fourier cosine coefficients \( p_n \) are nonnegative. For \( n = 0 \), this is evident from (iii). As a nonincreasing and convex function, \( \psi(t) \) has a right derivative \( \psi'(t) \) such that \( -\psi'(t) \) is nonnegative and nonincreasing on \([0,K]\). With condition (i) in mind, partial integration leads to

\[
p_n = \frac{1}{K} \int_{-K}^{K} \psi(t) \cos\left(\frac{\pi nt}{K}\right) \, dt
\]

\[
= \frac{2}{K} \int_{0}^{K} \psi(t) \cos\left(\frac{\pi nt}{K}\right) \, dt
\]
\[
\frac{2}{\pi n} \int_0^K [-\psi'(t)] \sin \left( \frac{n \pi t}{K} \right) \, dt \\
= \frac{2}{\pi n} \sum_{j=0}^n \int_{(j-1)K/n}^{jK/n} [-\psi'(t)] \sin \left( \frac{n \pi t}{K} \right) \, dt \geq 0
\]

for \( n = 1, 2, \ldots \), because the terms in the sum are alternating in sign and nonincreasing in absolute value. This concludes the proof. \( \square \)

The theorem is stronger than both Theorem 2.1 of Wood (1995) and Theorem 4.3.2 of Lukacs (1970). For example, Wood requires that \( \psi(t) \) be the truncation of a correlation function \( \phi(t) \) on the real line satisfying \( \phi'(t) \leq 0 \) and \( \phi''(t) \geq 0 \) for \( t > 0 \). Any such function is bounded and nonincreasing for \( t > 0 \) and must tend to some finite limit as \( t \) approaches infinity. It is easy to see that the limit, and therefore the function \( \phi(t) \), will be nonnegative. In contrast, Theorem 1 allows for functions \( \psi(t) \) that attain negative values. Such an extension is useful in a number of applications. Figs. 1 and 2 in the recent work of Dietrich (1995) on geostatistical simulation display correlation functions \( \phi(t) \) that stay convex for \( t \geq 0 \) until they reach a negative minimum for some \( t_0 > 0 \). Simulations of line processes with these correlation functions are essential for the proposed algorithm, and the positive results of Theorem 1 lend further support to Dietrich's recommendation of using the circulant embedding technique for the line process simulations. More detailed information on the circulant embedding approach and its connection to the present problem can be found in Dietrich and Newsam (1993) or Wood and Chan (1994).

3. A simple necessary condition

The following negative result is similar to Theorem 2.3 of Wood (1995), although both its assumptions and its proof are much simpler.

**Theorem 2.** Suppose \( \phi(t) \), \( t \in \mathbb{R} \), is a correlation function on the real line that has \( 2d \) derivatives at \( t = 0 \), where \( d \) is a positive integer. Let \( K > 0 \) and assume \( \phi^{(j)}(K) \neq 0 \) for some odd integer \( j, 1 \leq j \leq 2d - 1 \). Then \( \psi(t) \), defined by (3), is not a correlation function on the circle \([-K, K]\).

**Proof.** The proof is done by contradiction. Suppose \( \phi^{(j)}(K) \neq 0 \) for some odd integer \( j, 1 \leq j \leq 2d - 1 \). In case \( \psi(t) \), defined by (3), were a correlation function on the circle \([-K, K]\), we could extend this function to a \((2K)\)-periodic correlation function \( \phi_0(t) \) on the line. Then \( \phi_0(t) \) would have \( 2d \) derivatives at \( t = 0 \), but not at multiples of \( \pm K \). It is well known that such a behavior is impossible for a correlation function on the real line (Yaglom, 1987, pp. 66–67). \( \square \)

It is interesting to observe how Theorem 2 and Wood's negative result complement each other. If \( x \in (1, 2) \), the correlation function

\[
\phi(t) = \exp(-|t|^x), \quad t \in \mathbb{R},
\]

satisfies the conditions of Wood's Theorem 2.3 for \( d = 1 \) and every \( K > 0 \). Thus, \( \psi(t) \), defined by (3), is not a correlation function on the circle \([-K, K]\) for any \( K > 0 \). Nevertheless, Theorem 2 does not apply, because the correlation function (5) does not have a second derivative at \( t = 0 \).

On the other hand, consider the function

\[
\phi(t) = \frac{1}{2} \left( \exp(-t^2) + (1 + 3|t|(1 - |t|)^2) \right), \quad t \in \mathbb{R}.
\]
It is not difficult to show that (6) gives a correlation function on the real line with \( \varphi''(0) = -7 \) and \( \varphi'(t) \neq 0 \) for all \( t \neq 0 \) (compare Wendland, 1995, Table 1). Thus, \( \varphi(t) \) satisfies the assumptions of Theorem 2 for \( d = 1 \) and \( K > 0 \), and \( \psi(t) \), defined by (3), is not a correlation function on the circle \([-K,K]\) for any \( K > 0 \). However, Wood’s Theorem 2.3 does not apply if \( K > 1 \), because \( \varphi''(t) \) is discontinuous at \( t = 1 \).

As a final point in this section, we contrast the behavior of the function (6) to that of Wendland’s (1995) correlation function

\[
\varphi(t) = (1 + 3|t|)(1 - |t|)^3, \quad t \in \mathbb{R}.
\]

If \( K < 1 \), both Theorem 2 and Wood’s Theorem 2.3 show that \( \psi(t) \), defined by (3), is not a correlation function on the circle \([-K,K]\). Yet if \( K \geq 1 \) both criteria fail to apply, and Theorem 2.2 of Wood (1995) shows that \( \psi(t) \), given by (3), is a correlation function on the circle \([-K,K]\).

4. Further comments

The proof of Theorem 2 relies on the fact that for \( \varphi(t) \) a correlation function on the real line and \( k \) a positive even integer, existence of \( \varphi^{(k)}(0) \) imply existence of \( \varphi^{(k)}(t) \) for all \( t \). Thus, it is natural to ask the following question: If \( k \) is a positive odd integer, does existence of \( \varphi^{(k)}(0) \) imply existence of \( \varphi^{(k)}(t) \) for all \( t \)? A positive answer would evidently generalize Theorem 2, and it would show that condition (ii) in Theorem 2.3 of Wood (1995) is redundant. Yet the answer is in the negative. For each positive odd integer \( k \), Wolfe (1975) has an example of a periodic correlation function \( \varphi(t) \) such that \( \varphi^{(k)}(0) \) exists, but \( \varphi^{(k)}(t_m) \) does not exist for a sequence of positive numbers \( (t_m) \) where \( t_m \to 0 \) as \( m \to \infty \).

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