LINEAR MODELS FOR SPATIAL OR TEMPORAL MULTIVARIATE DATA

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ABSTRACT

In geostatistics, methods for characterizing the spatial or temporal variation at different scales of a multivariate system have attracted much attention during the last decade. Applications have been particularly numerous in soil science. Most work was based on the linear model of coregionalization (LCM) which is suitable for the description of a system using direct and cross variograms, because cross variograms are even. Recently, work done on complex kriging (for estimating vector variables in 2D geographical space) has inspired the bilinear model of coregionalization (BMC) which is more general in the sense that it allows the description of a system for which the cross-covariance functions are not even.

1 INTRODUCTION

Cross-covariance functions are useful in describing the cross-correlations in a set of variables which can be:

- different types of measurements located in space or time,
- measurements of one quantity in a spatial region at different times,
- measurements of one quantity along time at different sites of a spatial region.

The cross-covariance functions are not necessarily even as there can be various kinds of delay or shift effects at different characteristic space or time scales between the variables. At each characteristic scale of index u we shall however assume that the space or time correlation is governed by one correlation function $\rho_u(h)$.

The real linear model of coregionalization has the limitation that it can only serve to model a set of covariance functions in which the cross-covariance functions are even. It is necessary to introduce a complex linear model of coregionalization and to take its real part, the bilinear model of coregionalization, when the cross-covariance functions in a real covariance function matrix are not even.

In geostatistics, the estimation of two dimensional vector variables as complex variables was studied by Lajaunie and Béjaoui (1991), who examined ways of modeling the complex covariance function. This work inspired Grzebyk (1993) in formulating the bilinear model of coregionalization.

2 COVARIANCE FUNCTION MATRIX

We denote $C_{ij}(h)$ the cross-covariance function between two jointly second-order stationary random functions $Z_i(x)$ and $Z_j(x)$, where $x$ is the vector of the coordinates of a point in space or time, $h$ is a vector linking a pair of points in space or time, $Z$ is a real or complex random variable, $i$ and $j$ are indices of a set of $N$ random functions.

The matrix $C_{ij}(h)$ of direct and cross covariance functions for a given set of random functions is characterized by Cramér’s generalization of the Bochner-Khintchine theorem (Cramér, 1940).

3 INTRINSIC CORRELATION MODEL

The simplest model for real random functions is the following

$$C(h) = V \rho(h)$$

where $V$ is the matrix of variances and covariances $\sigma_{ij}$ and $\rho(h)$ is a direct correlation function.

It is called the intrinsic correlation model (Matheron, 1965) because the correlation between two random functions

$$\frac{\sigma_{ij} \rho(h)}{\sqrt{\sigma_{ii} \rho(h)} \sqrt{\sigma_{jj} \rho(h)}} = \frac{\rho_{ij}}{\sqrt{\sigma_{ii} \sigma_{jj}}} = \rho_{ij}$$

(2)

does not depend upon spatial scale.

The linear model associated to the intrinsic correlation model is written

$$Z_i(x) = \sum_{p=1}^{N} a_{ip} Y_p(x)$$

(3)

where $Y_p(x)$ are $N$ uncorrelated random functions whose direct covariance functions $\rho(h)$ do...
not depend on the index \( p \) and \( a_{pi} \) are transformation coefficients.

From a known intrinsic correlation model, one possible method to specify the transformation coefficients is based on the eigenvalue decomposition of the variance-covariance matrix \( \mathbf{V} \) and the factors \( Y_p \) can then be interpreted as principal components.

The intrinsic correlation model is an important reference case when the variables are all measured at the same locations, because their estimation is simplified (Wackernagel, 1995).

4 NESTED MODEL

A more sophisticated model for a set of real random functions is the multivariate nested covariance function model

\[
\mathbf{C}(\mathbf{h}) = \sum_{u=0}^{S} \mathbf{B}_u \rho_u(\mathbf{h})
\]

(4)

where \( u \) is an index for a set of \( S+1 \) characteristic spatial or temporal scales and the coregionalization matrices \( \mathbf{B}_u \) are variance-covariance matrices describing multivariate correlation at these characteristic scales of the phenomenon.

The associated random function model is the \textit{linear model of coregionalization} (LMC)

\[
Z_i(\mathbf{x}) = \sum_{u=0}^{S} \sum_{p=1}^{N} a_{pu} Y_{pu}(\mathbf{x})
\]

(5)

where a set of \( N \) uncorrelated factors is defined at each of the \( S+1 \) characteristic scales. A possibility to specify the LMC from a known multivariate nested covariance function model is by performing a principal component analysis based on the eigenvalue decomposition of the coregionalization matrices which yields the transformation coefficients \( Y_{pu}(\mathbf{x}) \).

5 COMPLEX COVARIANCE FUNCTION

A model for univariate complex covariance function \( C(\mathbf{h}) = C^{\text{Re}}(\mathbf{h}) + i C^{\text{Im}}(\mathbf{h}) \) with a known real part \( C^{\text{Re}}(\mathbf{h}) \) can be defined on the basis of an application of the Radon-Nikodym theorem by the relation

\[
i C^{\text{Im}}(\mathbf{h}) = (\Phi * C^{\text{Re}})(\mathbf{h})
\]

(6)

where \( \Phi \) is a complex distribution whose Fourier transform is a real odd function \( \varphi \) with values in the interval \([-1, 1]\). The class of \( C^{\text{Im}}(\mathbf{h}) \) corresponding to a given real continuous covariance function \( C^{\text{Re}}(\mathbf{h}) \), such that \( C(\mathbf{h}) = C^{\text{Re}}(\mathbf{h}) + (\Phi * C^{\text{Re}})(\mathbf{h}) \) is a complex covariance function, are called \textit{compatible} imaginary parts.

A simple class of compatible imaginary parts can be obtained using \( \Phi = i/2 (\nu - \overline{\nu}) \) where \( \nu \) is a bounded real measure such that \( |\int \varphi(\mathbf{u})| < 1 \) with \( \varphi(\mathbf{u}) = -\int \sin(\mathbf{u}^T \tau) \nu(d\tau) \). The compatible imaginary parts are given by

\[
C^{\text{Im}}(\mathbf{h}) = \frac{1}{2} \int [C^{\text{Re}}(\mathbf{h} - \tau_k) - C^{\text{Re}}(\mathbf{h} + \tau_k)] \nu(d\tau)
\]

The class can be extended (Grzegory, 1993) by considering real covariance functions \( C^{\text{Re}}(\mathbf{h}) \) compatible with a given \( C^{\text{Re}}(\mathbf{h}) \) in the sense that the positive measures \( \mu^c \leq \mu^{Re} \) and then

\[
C^{\text{Im}}(\mathbf{h}) = \frac{1}{2} \int [C^{\text{Re}}(\mathbf{h} - \tau_k) - C^{\text{Re}}(\mathbf{h} + \tau_k)] \nu(d\tau)
\]

is a compatible imaginary part, provided that the real bounded measure \( \nu \) satisfies \( |\int \sin(\mathbf{u}^T \tau) \nu(d\tau)| < 1 \).

In practice a finite sum based on translations \( \tau_k \) is used

\[
C^{\text{Im}}(\mathbf{h}) = \frac{1}{2} \sum_{k=1}^{K} \rho_k (C^{\text{Re}}(\mathbf{h} - \tau_k) - C^{\text{Re}}(\mathbf{h} + \tau_k))
\]

(8)

with weights \( \rho_k \geq 0 \) and \( \sum \rho_k \leq 1 \). Instead of only one function \( C^{\text{Re}}(\mathbf{h}) \), \( K \) functions \( C^{\text{Re}}_k(\mathbf{h}) \) can be introduced which represent a family of covariance functions compatible with a given \( C^{\text{Re}}(\mathbf{h}) \).

6 BILINEAR MODEL

The complex analogue to the intrinsic correlation model is

\[
\mathbf{C}(\mathbf{h}) = \mathbf{B} \rho(\mathbf{h}) = \mathbf{E} \chi(\mathbf{h}) - \mathbf{F} \kappa(\mathbf{h}) + i (\mathbf{E} \kappa(\mathbf{h}) + \mathbf{F} \chi(\mathbf{h}))
\]

(9)

where \( \rho(\mathbf{h}) = \chi(\mathbf{h}) + i \kappa(\mathbf{h}) \) is a scalar complex covariance function and \( \mathbf{B} \) is a hermitian positive semi-definite matrix with \( \mathbf{B} = \mathbf{E} + i \mathbf{F} \). The matrix \( \mathbf{E} \) is a symmetric positive semi-definite matrix while \( \mathbf{F} \) is antisymmetric.

Naturally we can consider a nested complex multivariate covariance function model of the type

\[
\mathbf{C}(\mathbf{h}) = \sum_u \mathbf{B}_u \rho_u(\mathbf{h})
\]

with a corresponding complex LMC.

The real LMC is in particular not adequate for multivariate time series analysis where delay effects or phase shifts are common and cannot be included in a model with even cross covariances. A model for real random functions with non even real cross covariance functions can be derived from the complex LMC by taking its real part. We obtain the \textit{bilinear model of coregionalization} (BMC) which is composed of the
linear combination of two sets of factors $U_{pu}(x)$ and $V_{pu}(x)$ with two sets of transformation coefficients $c_{piu}$ and $d_{piu}$.

In the case of only one spatial scale (the nested case is analog) we can drop the index $u$ and we have the BMC

$$Z_t(x) = \sum_{p=1}^{N} c_{pi} U_p(x) - \sum_{p=1}^{N} d_{pi} V_p(x) \quad (10)$$

with

$$\sum_{p=1}^{N} (c_{pi} c_{pi} + d_{pi} d_{pi}) = e_{ij} \quad (11)$$

$$\sum_{p=1}^{N} (c_{pi} d_{pi} - c_{pi} d_{pi}) = f_{ij} \quad (12)$$

where $e_{ij}$ and $f_{ij}$ are respectively the elements of matrices $E$ and $F$ such that $B = E + i F$.

Restraining the covariance functions for $U(x)$ and $V(x)$ to be of the form

$$C_{UV}(h) = C_{VV}(h) = \frac{1}{2} \chi(h) \quad (13)$$

$$C_{UV}(-h) = -C_{UV}(h) = \frac{1}{2} \kappa(h) \quad (14)$$

where $\chi(h) + i \kappa(h)$ is a complex covariance function with an odd imaginary part, the cross-covariance function between two real variables is

$$C_{ij}(h) = \sum_{p=1}^{N} \left( c_{ip}^2 d_{p}^2 + d_{p}^2 c_{ip}^2 \right) \frac{\chi(h)}{2} - \sum_{p=1}^{N} \left( c_{ip} d_{ip} - c_{ip} d_{ip} \right) \frac{\kappa(h)}{2} \quad (15)$$

The multivariate covariance function model associated to the BMC is thus

$$C(h) = \frac{1}{2} \left( E \chi(h) - F \kappa(h) \right) \quad (16)$$

and is real. See Grzebyk (1993) for details on fitting algorithms.

7 APPLICATIONS

Grzebyk (1993) provides an example of the fit of a covariance model with non even cross covariance functions using a BMC to the corregionalization of data from three remote sensing channels of a Landsat satellite.

The BMC certainly has many applications in meteorology. It can for example serve in modeling wind and geopotential (like in Chauvet et al. (1976) or Thiébaux et al. (1990)) because the extension of the BMC to the case of a set of variables including their first order derivatives is straightforward (Grzebyk and Wackernagel, 1994).

REFERENCES


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